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a. REPORT	b. ABSTRACT	c. THIS PAGE			I. R. Goodman
U	U	U	UU	9	19b. TELEPHONE NUMBER (Include area code) (619) 553-3680

# A NEW METHOD FOR REPRESENTING LINGUISTIC QUANTIFICATIONS BY RANDOM SETS WITH APPLICATIONS TO TRACKING AND DATA FUSION

*William C. Torrez*  
CNO STRATEGIC STUDIES GROUP,  
NAVAL WAR COLLEGE

Newport, Rhode Island 02841  
<mailto:torrezw@nwc.navy.mil>

*Donald Bamber and I. R. Goodman*  
CODE 244210,  
SPACE AND NAVAL WARFARE  
SYSTEMS CENTER,  
San Diego, California 92152  
<mailto:bamber@spawar.navy.mil>  
<mailto:goodman@spawar.navy.mil>

*Hung T. Nguyen*  
DEPARTMENT OF MATHEMATICAL SCIENCES,  
NEW MEXICO STATE UNIVERSITY  
Las Cruces, New Mexico 88003-8001  
<mailto:hunguyen@nmsu.edu>

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*William C. Torrez*  
CNO STRATEGIC STUDIES GROUP,  
NAVAL WAR COLLEGE  
Newport, Rhode Island 02841  
<mailto:torrezw@nwc.navy.mil>

*Donald Bamber and I. R. Goodman*  
SPACE AND NAVAL WARFARE SYSTEMS CENTER,  
CODE 244210  
San Diego, California 92152  
<mailto:bamber@spawar.navy.mil>  
<mailto:goodman@spawar.navy.mil>

*Hung T. Nguyen*  
DEPARTMENT OF MATHEMATICAL SCIENCES,  
NEW MEXICO STATE UNIVERSITY  
Las Cruces, New Mexico 88003-8001  
<mailto:hunguyen@nmsu.edu>

**Abstract** - *There is an obvious need to be able to integrate both linguistic-based and stochastic-based input information in data fusion. In particular, this need is critical in addressing problems of track association, including cyber-state intrusions. This paper treats this issue through a new insight into how three apparently distinct mathematical tools can be combined: "boolean relational event algebra" (BREA), "one point random set coverage representations of fuzzy sets" (OPRSC), and "complexity-reducing algorithm for near optimal fusion" (CRANOF).*

**Keywords** - fuzzy logic, random sets, second order probability, reasoning, tracking & correlation, data fusion

## 1. Introduction

This paper can be considered a direct extension and modification of ideas originally proffered in [12, 13, 24].

At the most general level, linguistic-based or natural language information is as important as probabilistic or stochastic information. In track association and security and intrusion problems, this is certainly the case. In the paper presented here, we show how three mathematical tools can be utilized in tandem to address a large class of problems involving both linguistic-based and probabilistic-based information.

These mathematical tools considered in the above task are "boolean relational event algebra" (BREA), "one point random set coverage representations of fuzzy sets" (OPRSC), and "complexity-reducing algorithm for near optimal fusion" (CRANOF). All of these have previously been applied to extend the capabilities of traditional probability and logic in analyzing a number of real-world-based problems.

BREA [12-17] has been used, at least in theory, to address problems involving compositional functions of probabilities, including the special case of arithmetic division, leading to the ability to compare and contrast inference rules from an internal algebraic, as well as a numerical, viewpoint.

OPRSC [10, 12, 13, 15-17, 19], on the other hand, has been useful in connecting fuzzy logic concepts directly and rigorously with corresponding stochastic-based concepts, as well as motivating a number of new fuzzy logic definitions and relations.

A preliminary form of CRANOF [4, 5, 7, 8, 9-12, 24] has already shown how a number of well-known "gaps" between classical logic reasoning and natural appearing probabilistic counterparts can be closed through its second order, i.e., randomized, probability structure.



We first review briefly – with an eye toward motivations – BREA (in Section 2), where each sub-type corresponds to some particular numerical function (or common type of functions) with domain in the unit interval (or its n-fold form) and with range in the unit interval. We also briefly consider OPRSC and fuzzy logic concepts (Section 3). Next (Section 4), we also briefly describe CRANOF. Finally, in Section 5 an outline of a procedure is presented, demonstrating how BREA and OPRSC can be directly combined, so that a large class of fuzzy logic quantified statements can be fully integrated into a single probability setting, along with other initially modeled probabilistic information. In turn, this can be utilized as probability-based information inputs into CRANOF, yielding a test for the validity of the reasoning processes to be considered. It is also pointed out that this procedure can be specialized to problems of tracking and correlation of targets of interest having disparate attribute information present, as is typical in intelligence decision-making problems, such as shown in last year's special session [24].

## 2. Summary of BREA

### 2.1. Boolean conditional event algebra as a special case of BREA

Referring again to the references in [12-17], BREA originally began with the proper sub-concept of Boolean Conditional Event Algebra (BCEA) (see, in particular, [15], [16]). *One motivating factor behind the development of BCEA is the desire to be able to assign reliabilities to various logical combinations of rules (or conditional expressions or if-then statements, etc.), when each rule has an assigned reliability in the form of naturally corresponding conditional probability.* For example, consider the conjunction of "if b, then a", "if d, then c", i.e., "(if b then a) and (if d, then c)", where reliabilities have already been assigned to each rule separately, in the form of conditional probabilities,  $P(a|b)$ ,  $P(c|d)$ , respectively. Thus, from now on, using the symbol  $(a|b)$  to represent the rule "if b, then a", and noting that in general rules need not be perfect, with conditional probability represented as usual by  $P(a|b) = P(ab)/P(b)$ , provided  $P(b) > 0$ , where either  $\&$  or no symbol represents boolean conjunction and we denote ordinary events (or, equivalently, propositions) by lower case roman letters, etc., we wish to make sense of either

$$P((a|b)\&(c|d)) \quad (2.1)$$

or for some appropriate extensions  $P^*$  of  $P$  and  $\&^*$  of  $\&$ ,

$$P^*((a|b)\&^*(c|d)), \quad (2.2)$$

where the basic compatibility relations hold

$$P^*((a|b)) = P(a|b), \quad P^*((c|d)) = P(c|d). \quad (2.3)$$

Another motivating force leading to CEA is the desire to be able to weed out of rule-based systems those rules that appear contradictory or nearly contradictory, or the opposite: the same or nearly the same. Thus, we seek a measure of contrast / similarity for rules. In the case of ordinary events (in a common boolean algebra), say  $a$ ,  $c$ , some natural measures that also take into account the probability of  $a$ ,  $c$  are  $P(a\Delta c)$  and  $P(a\Delta c | a \vee c)$ , where  $\vee$  is boolean disjunction and  $\Delta$  is the boolean symmetric difference operation and  $(.)'$  is boolean negation. Thus,

$$\begin{aligned} a\Delta c &= ac' \vee a'c; \\ P(a\Delta c) &= P(ac') + P(a'c) = P(a) + P(c) - 2P(ac), \\ P(a\vee c) &= P(a) + P(c) - P(ac), \end{aligned} \quad (2.4)$$

yielding

$$P(a\Delta c | a \vee c) = (P(a) + P(c) - 2P(ac)) / (P(a) + P(c) - P(ac)). \quad (2.5)$$

See [15] for more details justifying that, indeed,  $P(a\Delta c)$  and  $P(a\Delta c | a \vee c)$  are actually legitimate pseudometrics in the mathematical sense with respect to  $a$ ,  $c$ .

Suppose that the following relations hold, denoted as

*Assumption I:*

Let  $B$  be a boolean algebra containing  $a$ ,  $b$ ,  $c$ ,  $d$ ,... here and  $P: B \rightarrow [0,1]$  a probability measure over  $B$ . Suppose there are extensions (in some natural sense)  $B^*$  of  $B$  and  $P^*$  of  $P$ , where  $B^*$  is a boolean algebra and  $P^*: B^* \rightarrow [0,1]$  is also a probability measure. Suppose also all rules formed from ordinary events  $a$ ,  $b$ ,  $c$ ,  $d$ ,... in  $B$  -- whence  $(a|b)$ ,  $(c|d)$  -- are in  $B^*$ . Hence, all logical combinations of them via the boolean operations of  $B^*$ ,  $\&^*$ ,  $\vee^*$ ,  $(.)'$ , etc., are still in  $B^*$ .

Then, e.g., by replacing  $a$  by  $(a|b)$ ,  $c$  by  $(c|d)$ ,  $\&$  by  $\&^*$ ,  $\vee$ ,  $\vee^*$ , etc., in eq.(2.5), still assuming also the compatibility relations of eq.(2.3),

$$\begin{aligned} &P^*((a|b)\Delta^*(c|d) | (a|b)\vee^*(c|d)) \\ &= [P(a|b) + P(c|d) - 2P^*((a|b)\&^*(c|d))] / \\ &[P(a|b) + P(c|d) - P^*((a|b)\&^*(c|d))]. \end{aligned} \quad (2.6)$$

$P^*((a|b)\Delta^*(c|d) | (a|b)\vee^*(c|d))$  in eq.(2.6) can be evaluated, *provided*  $P^*((a|b)\&^*(c|d))$  can be obtained.  $P^*((a|b)\Delta^*(c|d) | (a|b)\vee^*(c|d))$  can then be used as a reasonably measure of contrast / similarity: the smaller its value the more similar  $(a|b)$  is to  $(c|d)$  with respect to  $P$ .

Furthermore, if distributional considerations can be introduced, where the probability measure  $P$  itself is allowed to vary in some way, so that  $P^*((a|b)\Delta^*(c|d) | (a|b)\vee^*(c|d))$  in eq.(2.6) becomes a random variable, say  $X$ ,

and if the cumulative distribution  $F$  of  $X$ , given  $H_0$  holds is obtainable, then the computation of the pseudometric in eq.(2.6) yields, in a standard way, a test of hypotheses  $H_0$  vs.  $H_1$ , where

$H_0 =^d (a|b) \text{ and } (c|d) \text{ are sufficiently similar so as to be merged in some sense or just one of them kept in the system,}$  (2.7)

$H_1 =^d (a|b) \text{ and } (c|d) \text{ are not sufficiently similar; thus keep both in system, etc.,}$  (2.8)

For some standard significance test level  $\lambda$ ,  $0 < \lambda \ll 1$ , determining test threshold  $K_\lambda$  via type one error

$\lambda = \text{Prob}(\text{Reject } H_0 \mid H_0 \text{ holds}) = 1 - F(K_\lambda)$ ,  
whence,

$$K_\lambda = F^{-1}(1 - \lambda), \quad (2.9)$$

where one chooses  $H_0$  at significance level  $\lambda$

$$\text{iff } P^*((a|b)\Delta^*(c|d) \mid (a|b)\vee^*(c|d)) < K_\lambda; \quad (2.10)$$

or, one chooses  $H_1$  at significance level  $\lambda$

$$\text{iff } P^*((a|b)\Delta^*(c|d) \mid (a|b)\vee^*(c|d)) \geq K_\lambda. \quad (2.11)$$

(Again, see [15] for more details of a procedure somewhat similar to the above one.)

Fortunately, one can show that for any given boolean algebra  $B$  and probability  $P$  over  $B$ , Assumption I can always be satisfied. For details of one natural approach, called the Product Space one, see, e.g., [13, 15, 16]. In this case, the extended measurable space  $(B^*, P^*)$  is called Product Space Conditional Event Algebra (PSCEA), due to its (countable) product space structure and the existence of rules as actual "conditional" events in  $B^*$ . In addition, it has been most useful to consider, in addition to the ordinary boolean operators ( $\&^*$ ,  $\vee^*$ ,  $(.)^*$ ) over  $B^*$ , certain modifications of non-boolean operators originally introduced independently by Adams [1,2] and Calabrese [8'], relative to spaces somewhat different in structure than PSCEA. In the case of Adams, these operators were used to characterize algebraically his *High Probability Logic* [1, 2] (see also Section 4 here). We denote these (modified) non-boolean operators over  $B^*$  as  $\&_{AC}$  and  $\vee_{AC}$ , to indicate their conjunctive-like and disjunctive-like properties of each, respectively.

## 2.2. Boolean weighted average event algebra as a special case of BREa.

Note that the realization of Assumption I in Section 2.1, together with the compatibility relation in eq.(2.3), shows that any BCEA, such as PSCEA, produces a homomorphic-like" (or "commutative diagram-like") relation, via probability (in the form of conditional probability) connecting proper numerical division (of

probabilities) over the unit interval (i.e., with the restriction that the numerator is less than or equal to the denominator, with the latter always positive) with an "algebraic division" in the form of the conditional event itself. Consider now, instead of arithmetic division of probabilities, the rather different weighted average operation. Let  $w =^d (w_1, \dots, w_n)$  be a fixed weight vector of real numbers, where each  $0 < w_j < 1$  and  $w_1 + \dots + w_n = 1$ , and define the averaging function  $f_w: [0,1]^n \rightarrow [0,1]$  by

$$f_w(x) =^d w_1 x_1 + \dots + w_n x_n, \quad (2.12)$$

where  $x = (x_1, \dots, x_n)$  in  $[0,1]^n$ , i.e.,  $0 \leq x_j \leq 1$ . Next, considering any  $n$  events  $a_1, \dots, a_n$  in  $B$ , and the relative atoms generated (by either use of conjunctions as we show here, together with evaluations via  $P$ , or alternatively, using cartesian products and product probability based upon identical marginal probability  $P$ ), indicated typically as

$$\omega_g =^d (a_1)^{[g(1)]} \& \dots \& (a_n)^{[g(n)]},$$

where  $g: \{1, \dots, n\} \rightarrow \{0,1\}$  is arbitrary and for any  $c$  in  $B$ ,

$$c^{[0]} =^d c, \quad c^{[1]} =^d c' \quad (2.13)$$

Then, it is easily proven that, using the atomic decomposition of each  $a_j$  as the disjunction of all relative atoms in which only  $a_j$  - not,  $a_j'$  - appears as a conjunctive factor,

$$\begin{aligned} & w_1 P(a_1) + \dots + w_n P(a_n) \\ &= \sum_{(\text{all possible } g: \{1, \dots, n\} \rightarrow \{0,1\}, \text{ except for } g_0)} (P(\omega_g) \cdot W(g)) \end{aligned} \quad (2.14)$$

where

$$g_0(k) =^d 1, \text{ for } k = 1, \dots, n, \quad (2.15)$$

and

$$W(g) =^d \sum_{(1 \leq k \leq n, g(k) = 0)} (w_k). \quad (2.16)$$

In turn, eq.(2.14) suggests that one could choose here (among possibly many other approaches) for the algebraic counterpart to  $f_w$ , using the same function symbol,  $f_w: B_n \rightarrow B \times [0,1]$ , where  $[0,1]$  is endowed with the usual Borel field of subsets, and for any  $a_1, \dots, a_n$  in  $B$ ,

$$f_w(a_1, \dots, a_n) =^d \bigvee_{(\text{all possible } g: \{1, \dots, n\} \rightarrow \{0,1\}, \text{ except for } g_0)} (\omega_g \times [0, W(g)]). \quad (2.17)$$

In line with the terms "conditional event" and "conditional event algebra", as used in Section 2.1, we can call any  $f_w(a_1, \dots, a_n)$  a weighted average event and  $(B^*, P^*)$  here a boolean weighted average algebra.

Then, for any given probability measure  $P$  over  $B$ , define probability measure  $P^*$  over the boolean algebra spanned by  $B \times [0,1]$ , i.e., set-wise, the collection of all



finite disjunction of products  $c \times A$ , for any  $c$  in  $B$  and any  $A$  in the borel field over  $[0,1]$ : First, define  $P^*$  as the restriction of the simple product probability measure of  $P$  with the uniform or lebesgue probability measure over  $[0,1]$ , and then extend this in the obvious way. This finally yields the analogue of the compatibility condition in eq.(2.3), for all  $a_j$  in  $B$  and all  $P$  over  $B$ ,

$$P^*(f_w(a_1, \dots, a_n)) = f_w(P(a_1), \dots, P(a_n)). \quad (2.18)$$

Note here, that, unlike the role that the single operation of arithmetic division plays in conditional event algebra, here the class of all weighted average operations  $f_w$  (as  $w$  is made to vary) in general corresponds to the same choice of weighted average event algebra.

Again, we can ask, analogous to the motivation for introduction of conditional events, how can the relation in eq.(2.18) be used? As one response to this, consider, in place of the issue of determining similarity or contrast between two rules of interest, as discussed in Section 2.1, the issue of comparing also for similarity or difference, the method of modeling of two experts, where each averages the input probabilities corresponding to several factors by supplying, in general different weighting factors. Of course, such experts could use other, far more complicated ways of fusing together such available probabilities, but for simplicity, we restricted the available information to be in the form of probabilities of separate attributes and the fusion procedure here to be in the form of simple weighted averaging of the probabilities.

Thus, for example, one could consider the problem of estimating the probability of an intrusion somewhere in a given complex system, when the only information available (from appropriate intelligence sources) are commonly known probabilities of the intrusion being directed by group  $Q$  and the intrusion occurs in area  $R$ . Suppose then each expert averages the two available probabilities in different ways, according to his / her own prior general knowledge, bias, etc. Thus, in effect we wish to test or determine for similarity or difference

$$\begin{aligned} \text{Expert 1: } & (1/2) \cdot P(a) + (1/2) \cdot P(b) \\ \text{vs.} & \\ \text{Expert 2: } & (1/3) \cdot P(a) + (2/3) \cdot P(b). \end{aligned} \quad (2.19)$$

Then, without going into details (see [12] for this same example with full analysis; see also, e.g., [14, 15, 17] for more general analysis), one can use the natural consistency relations in eq.(2.18), analogous to the estimation and testing of hypotheses procedure for rules in Section 2.1. For the general case with weighting vectors  $w(j) = (w_{j,1}, \dots, w_{j,n})$ ,  $j = 1, 2$

$$\text{Expert 1: } w_{1,1} \cdot P(a_1) + \dots + w_{1,n} \cdot P(a_n)$$

vs.

$$\text{Expert 2: } w_{2,1} \cdot P(a_1) + \dots + w_{2,n} \cdot P(a_n), \quad (2.20)$$

where

$$f_{w(j)}(x_1, \dots, x_n) = {}^d w_{j,1} \cdot x_1 + \dots + w_{j,n} \cdot x_n, \quad (2.21)$$

all of this depends on the evaluation of the key quantity

$$\begin{aligned} & P^*(f_{w(1)}(a_1, \dots, a_n) \Delta^* f_{w(2)}(a_1, \dots, a_n) \mid \\ & \quad f_{w(1)}(a_1, \dots, a_n) \vee^* f_{w(2)}(a_1, \dots, a_n)) \\ &= [f_{w(1)}(P(a_1), \dots, P(a_n)) + f_{w(2)}(P(a_1), \dots, P(a_n)) \\ & \quad - 2P^*(f_{w(1)}(a_1, \dots, a_n) \&^* f_{w(2)}(a_1, \dots, a_n))] / \\ & \quad [f_{w(1)}(P(a_1), \dots, P(a_n)) + f_{w(2)}(P(a_1), \dots, P(a_n)) \\ & \quad - P^*(f_{w(1)}(a_1, \dots, a_n) \&^* f_{w(2)}(a_1, \dots, a_n))]. \end{aligned} \quad (2.22)$$

Finally, it should be pointed out that it can also be demonstrated in a straightforward way that the approach to comparison via eq.(2.22) is superior in a number of ways to the naive approach that simply compares externally the two models numerically, i.e., utilizes simply

$$|f_{w(1)}(P(a_1), \dots, P(a_n)) - f_{w(2)}(P(a_1), \dots, P(a_n))|. \quad (2.23)$$

### 2.3. Boolean single argument function algebra as a special case of BREa.

Next, consider any single argument function  $g: [0,1] \rightarrow [0,1]$ . In (first order) fuzzy set parlance [22],  $g$  is not only a fuzzy set membership function, but is a typical (external) type of fuzzy set modifier. For example if  $g$  is some monotone increasing function with  $g(j) = j$ ,  $j = 0, 1$ , depending upon its shape,  $g$  can be naturally identified with a corresponding linguistic quantifier, such as, e.g., when  $g(x) = x^2$ , for any  $0 \leq x \leq 1$ ,  $g$  is often called "very", while, if  $g(x) = x^{1/2}$ , for any  $0 \leq x \leq 1$ ,  $g$  is often called "somewhat", etc. Or, if  $g$  is decreasing, with  $g(j) = 1-j$ , for  $j = 0, 1$ ,  $g$  is usually identified as a negative type of quantifier, etc.

Hence, in this context, for any such  $g$ , one seeks the extensions  $B^*$  of  $B$ ,  $P^*$  of  $P$  over  $B$ , with  $P^*$  a probability measure over  $B^*$ , so that  $g$  as an algebraic function  $g: B \rightarrow B^*$ , for any  $a$  in  $B$ , again the analogue of eq.(2.3), is sought:

$$P^*(g(a)) = g(P(a)). \quad (2.24)$$

If  $g(x)$  is a polynomial or is analytic in  $x$  (i.e., an infinite series) with coefficients in  $[0,1]$ , so that its range is also in  $[0,1]$  for all  $x$  in  $[0,1]$ , then corresponding  $g$ -events have been determined [13, 14, 17]

### 3. Summary of OPRSC

OPRSC was developed in order to represent in a natural way (first order) fuzzy sets and fuzzy logic concepts by corresponding probability ones. This provides a general insight into already-developed fuzzy logic notions and an overall guideline for the potential development of new fuzzy logic concepts. Basic references here include [10, 13, 17, 19, 21]. First, recall that given any fuzzy set membership function  $h:D \rightarrow [0,1]$ , for some domain  $D$ , in general there exist a large class of distinct (but by no means, all) random subsets  $S(h, \text{cop})$  of  $D$ , indexed by the class of all copulas  $\text{cop}$  with arguments corresponding to elements of  $D$ , that are one-point coverage-equivalent to  $h$ , with corresponding probability measure  $P_{h,\text{cop}}$ ,

$$P_{h,\text{cop}}(x \text{ in } S(h, \text{cop})) = h(x), \text{ all } x \text{ in } D. \quad (3.1)$$

(See [21] for background on copulas, co-copulas, and survival copulas  $\text{cop}^\wedge$ .) Eq.(3.1) shows fuzzy set membership functions are a sort of weak specification of certain classes of random subsets of their domains, much as the expectation provides weak information about the random variables it represents.

In addition, a number of "homomorphic-like" relations hold with established fuzzy logic concepts, such as the "fuzzy extension principle" or "fuzzy conditioning" for populations, using ratios of "fuzzy cardinalities" of attributes. For the latter concept, consider for say,  $g, h: D \rightarrow [0,1]$ ,  $w:D \rightarrow [0,1]$ ,  $D$  finite with  $w$  a weighting vector, the *weighted population (over  $D$ ) averaged conditioning of attribute  $g$  to  $h$*

(3.2)

$$(g|h)_{\text{cop},w} = \frac{\sum_{x \text{ in } D} (w(x) \cdot \text{cop}^\wedge(g(x), h(x)))}{\sum_{x \text{ in } D} (w(x) \cdot h(x))},$$

where  $\text{cop}^\wedge$  is the survival copula associated with copula  $\text{cop}$ , defined for any  $y, z$  in  $[0,1]$  as

$$\text{cop}^\wedge(y,z) = y + z - \text{cop}(y,z), \quad (3.3)$$

where in [21], (radial symmetry) conditions are established for copulas to coincide with their survival copula. Then, it follows [10] that

$$\begin{aligned} (g|h)_{\text{cop},w} &= P_{g,\text{cop}}(a_{g,\text{cop}} | b_{g,\text{cop}}) \\ &= P_{g,\text{cop}}(V_w \text{ in } S(g,\text{cop}) | V_w \text{ in } S(h,\text{cop})), \end{aligned} \quad (3.4)$$

where events

$$a_{g,\text{cop}} = \{V_w \text{ in } S(g,\text{cop})\}, \quad a_{h,\text{cop}} = \{V_w \text{ in } S(h,\text{cop})\}; \quad (3.5)$$

$V_w$  is a random variable over  $D$  with probability function  $w$ , independent of random subsets  $S(g,\text{cop})$ ,  $S(h,\text{cop})$  of  $D$ ,

the latter being one-point coverage equivalent to  $g$ ,  $h$ , respectively.

### 4. Brief summary of CRANOF.

Choose a collection of rules  $(a_j|b_j)$ ,  $j$  in  $J$ ,  $a_j, b_j$  in  $B$ ,  $J$  a finite index set, labeling  $(a|b)_J = \{(a_j|b_j)_{j \text{ in } J}\}$  as a *premise set*, and choose another rule of interest  $(c|d)$ , labeled as a *potential conclusion*, and call  $G = \{(a|b)_J; (c|d)\}$  an *entailment scheme* of interest, in which we wish to determine the behavior of  $P(c|d)$ , when the quantity  $P(a|b)_J = P(a_j|b_j)_{j \text{ in } J}$  is either known or certain restrictions on these values are known. A typical (lower bound threshold) form of the *underconstrained probability problem* for  $G$  can be phrased as

$$\text{Given: } P(a_1|b_1) \geq t_1, \dots, P(a_m|b_m) \geq t_m; \quad (4.1)$$

$$\text{Estimate in some best sense } P(c|d), \quad (4.2)$$

where the  $t_j$  are known, as are the events  $a_j, b_j, c, d$ , but  $P$  is not, up to satisfying the above constraints. In light of Section 2, the above can be interpreted in terms of reliabilities of rules.

Because in general the solution of eqs.(4.1 and (4.2) is difficult, a typical approach to this issue is to consider the corresponding "high probability" version of eq.(4.1), where now the thresholds  $t_j$  are allowed to vary toward unity in certain specified ways (such as uniformly or via a fixed "scaled" rate) and one then determines whether correspondingly the probabilities satisfying these constraints produce a limit that is either unity – the "valid" case for  $G$  – or less than unity – the "invalid" case for  $G$ . (Or, correspondingly, we can say  $(a|b)_J$  entails  $(c|d)$ , in some appropriate sense, or that  $(a|b)_J$  does not entail  $(c|d)$  in the same sense.)

Prominent among those taking a high probability approach with a uniform rate of limit has been Adams [1, 2] (with further analysis provided in [5, 9]) in fact this approach has been universally known simply as High Probability Logic (HPL), with associated HPL-validity or HPL-invalidity holding relative to  $G$ . Succinctly, this can be summarized as simply

$$\begin{aligned} G \text{ is HPL-valid} &\text{ iff } \lim_{(t_j \uparrow 1 \text{ uniformly})} (\minconc(G)(t_j)) = 1, \\ &\quad (4.3) \end{aligned}$$

$$G \text{ is HPL-invalid} \text{ iff } \lim_{(t_j \uparrow 1 \text{ uniformly})} (\minconc(G)(t_j)) < 1,$$

where the *minimum conclusion function* of  $G$  is defined as

$$\minconc(G)(t_j)$$



$$=_{\text{d}} \inf\{P(c|d): \text{all } P \text{ over } B \text{ such that eq.(4.2 holds)}\}. \quad (4.4)$$

Many patterned special cases of eq.(4.2) can also be considered either as extensions of valid or invalid Classical Logic entailment schemes, or arising from AI considerations. (See, e.g., [20, 23] for more details.) These include such well-known schemes as *transitivity*, *contraposition*, *strengthening of antecedent*, and *positive conjunction*, among many others. However, it turns out that

$$G \text{ is HPL-invalid iff } \lim_{(t_j \uparrow 1 \text{ uniformly})} (\minconc(G)(t_j)) = 0. \quad (4.5)$$

Thus, the minconc function of  $G$  cannot be a nontrivial measure of the degree of validity of  $G$  in the HPL sense. Moreover, it also follows that many patterned entailment schemes, commonly agreed by the AI / Rule-Based communities [20, 23] that ought to be valid in some sense, in agreement with commonsense reasoning, fail to be HPL-valid, including all four of the above-mentioned special schemes.

The independent work of Bamber [3, 4] and that of Goodman [5] and Goodman & Nguyen [18], utilize, in one sense or another, second order probability – i.e., probability of probabilities, where the latter are assigned some prior distribution. This can be relative to the fixed threshold problem in eqs.(4.1) and (4.2) (or related to such) or the high probability version of such. This work was followed by the joint efforts of Bamber, Goodman, and Nguyen in [5-8, 11, 12, 24], leading to the CRANOF algorithm that determines both the direct estimation and high probability version of eqs.(4.1) and (4.2). Roughly speaking, CRANOF utilizes in place of Adams HPL criterion function, minconc( $G$ ), the more moderating criterion function meanconc( $G$ ). Here, using obvious multivariable notation,

$$\text{meanconc}(G) =_{\text{d}} E_P(P(c|d) \mid P(a|b)_j \geq t_j), \quad (4.6)$$

where some appropriate prior distribution is assigned to  $P$  now considered as a random quantity. In general, the earlier assumption that the prior should be uniform is replaced by an arbitrary, but fixed, Dirichlet distribution (the Dirichlet family includes the uniform distribution as a special case and can be justified from several viewpoints as “best” prior [18]). Again, because of the possibility of complex calculations arising, a modified version of meanconc( $G$ ) has also been considered in “plug-in” form:

$$P^{\wedge}(c|d); \quad P^{\wedge} =_{\text{d}} E_P(P(\cdot) \mid P(a|b)_j \geq t_j), \quad (4.7)$$

whence, by basic linear properties of expectations, etc.,

$$P^{\wedge}(c|d) = P^{\wedge}(cd) / (P^{\wedge}(cd) + P^{\wedge}(c'd)), \quad (4.8)$$

$$P^{\wedge}(cd) = E_P(P(cd) \mid P(a|b)_j \geq t_j), \quad (4.9)$$

$$P^{\wedge}(c'd) = E_P(P(cd) \mid P(a|b)_j \geq t_j) \quad (4.10)$$

Most recently in [11] a full closed-form evaluation of the (fixed threshold) plug-in version in eqs.(4.7)-(4.10) has been obtained (for the general Dirichlet prior assumption for  $P$ ), that also appears feasible to implement. Moreover, this solution is directly related to Bamber's original uniform limit-directed high probability logic [4], dubbed by him as the “Unscaled” Near Surety Probability Logic UNSPL criterion for validity / invalidity of any entailment scheme  $G$ . According to Bamber's original algebraic characterization in [4], this means that both the fixed threshold and high probability approach to  $G$ , in effect, have the original premise set  $(a|b)_j$  replaced by a much smaller class of rules. This smaller class arises as follows: First, an initial collection of sets of rules, labeled as “exclusion classes”, of the premise set of  $G$  is automatically determined. Each such set of rules is nested in the next set. Also, the smaller the nested set of rules, the “rarer” it is, compared to the entire premise set of rules. Then for each such class, a single rule, in effect, is automatically selected, thereby reducing significantly the premise set size. In [11] it is pointed out that this single rule is actually the  $\&_{AC}$  operation over the corresponding exclusion class of rules in material conditional form; it is also indicated that Bamber's algebraic characterization is equivalent to a modified form of material conditional logic, that accounts for ambiguous validity relative to certain of the exclusion classes, where the potential conclusion rule is also replaced by its corresponding material conditional form.

## 5. Combining BREa and OPRSC.

This section shows one way in which BREa and OPRSC can be combined to treat many fuzzy logic quantified expressions. Consider first any given fuzzy set membership function  $h:D \rightarrow [0,1]$ , as before, and some external  $g$ -modifier of  $h$  as discussed in Section 2.3, where  $g:[0,1] \rightarrow [0,1]$ . If for each  $a$  in  $B$  a  $g$ -event  $g(a)$  exists in the sense of relational event algebra, i.e., there is some boolean algebra  $B_g^*$  extending  $B$  and a probability measure  $P_g^*$  over  $B_g^*$ , extending  $P$ , such that eq.(2.24) holds. In particular, in that equation, choose  $a$  to be  $a_{h,cop,x}$ , where event

$$a_{h,cop,x} =_{\text{d}} (x \text{ in } S(h,cop)), \quad (5.1)$$

where random subset  $S(h,cop)$  is as before. Hence, eq.(2.24) implies, for any copula  $cop$  indexed by  $D$ ,

$$g(h(x)) = (P_{h,cop})_g^*(g(a_{h,cop,x})), \text{ for all } x \text{ in } D. \quad (5.2)$$

The relation in eq.(5.2), of course can be extended to  $g$  having multiple arguments, again provided that the



corresponding form of (2.24) holds. For simplicity, suppose all of the functions  $g_j$  considered here, are single argument modifiers, i.e.,  $g_j: D_j \rightarrow [0,1]$ , and  $h_j: D_j \rightarrow [0,1]$  is a fuzzy set membership function,  $j = 1, 2, \dots, m$ , so that, analogous to the premise set information of eq.(4.1) given in lower bound threshold form, we have

$$(g_1(h(x_1) \geq t_1), \dots, (g_m(h(x_m) \geq t_m)) \quad (5.3)$$

In a natural sense, the composition function values  $g_j(h_j(x_j))$  correspond to linguistic information in the form of  $g_j$ -modified degree of attribute  $j$  possession, for  $j = 1, \dots, m$ . In addition, information may also be present in direct probability form, provided also as originally in eq.(4.1) as, e.g.,

$$(P(b_1) \geq s_1), \dots, (P(b_n) \geq s_n), \quad (5.4)$$

for appropriately known events  $b_j$ . Of course, all of the  $s_i$  and  $t_j$  are known. Finally, suppose that we wish to estimate  $P(c)$  under the above knowledge. By use of eq.(5.2), the inequalities in eqs.(5.3) and (5.4) become

$$\begin{aligned} &((P_{h_1, \text{cop}})_{g_1} * (g_1(a_{h_1, \text{cop}, x_1})) \geq t_1), \dots, \\ &(((P_{h_m, \text{cop}})_{g_m} * (g_m(a_{h_1, \text{cop}, x_m})) \geq t_m)) \\ &\text{for all } x \text{ in } D, \text{ and} \\ &(P(b_1) \geq s_1), \dots, (P(b_n) \geq s_n), \end{aligned} \quad (5.5)$$

Thus, e.g., one can consider estimation of  $P(c)$  via both the linguistic and original probabilistic collections in eq.(5.5), in complete probability form, where, the original separate variable probability measures  $P$  and  $(P_{h_j, \text{cop}})_{g_j}^*$  are replaced by a common variable joint probability measure. This is clearly an unconditional entailment problem.

Going one step further, such as illustrated in eq.(3.4) apropos to fuzzy logic conditioning of a population – whence, corresponding to linguistic conditioning of the population – and where the original probabilistic aspect of the information is also given in conditional form, but also with suitable modification of eq.(5.2), one then is faced essentially with a form of the rule-based entailment problem, which is treatable by use of CRANOF, as outlined in general in Section 4.

Lack of space here precludes any further details, except to say that for many types of tracking or fusion problems, the above procedure appears to be a reasonable method for combining linguistic-based and probabilistic-based information into a common entailment problem, analogous to the detailed approach provided in a previous application of CRANOF to tracking association issues [24].

## 6. Conclusions

Over the past number of years the co-authors of this paper have made contributions to the development of a number of mathematical tools that are useful in problems of track association and data fusion. This paper provides new insight into the meaning and use of these tools, as well as shows how three of them can be systematically combined to provide an approach to fusing both linguistic and probabilistic descriptions.

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